

Random walks on hyperplane arrangements and stopping times

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Abstract

Consider a real hyperplane arrangement and let \mathcal{C} denote the occurring chambers. Bidigare, Hanlon and Rockmore introduced a Markov chain on \mathcal{C} which is a generalization of some card shuffling models used in computer science, biology and card games. This paper introduces strong stationary arguments for this Markov chain, which provide explicit bounds for the separation distance.

1 Introduction

Consider the following process on a finite, transitive graph: pick a vertex at random and flip a fair coin to determine whether to color this vertex and its neighbors red or blue. Now, consider the following card shuffling scheme: enumerate the subsets of $\{1, 2, \dots, n\}$ and assign weight w_i to the i^{th} subset. Pick a subset of $\{1, 2, \dots, n\}$ according to w and move the cards indicated by that set to the top keeping their relative order. This is a generalization of the riffle shuffles, called the pop shuffles model. It turns out that these two processes are quite similar: they both are Markov processes on the chambers of some hyperplane arrangement.

A very special case of the second example is the Tsetlin library or (weighted) random to top card shuffling: consider a collection of books (or cards), labeled 1 through n . Pick a book i with probability w_i and move it to the front. This is a very well studied Markov chain mainly because of its use in dynamic file maintenance and cache maintenance ([12], [13], [15]). The eigenvalues of this process were discovered independently by Donnelly [12], Kapoor and Reingold [14], and Phatarfod [15] .

Most of the processes on graphs of this type are viewed as Markov chains on the chambers of the Boolean arrangement. The card shuffling schemes mentioned above are treated as Markov chains on the chambers of the braid arrangement. Examples of card shuffling, hypercube walks and coloring processes are studied thoroughly in Sections 3 and 4.

The unifying picture is the following: let \mathcal{A} be a finite collection of affine hyperplanes in $V = \mathbb{R}^n$ which is called a hyperplane arrangement. These hyperplanes cut V in finitely many connected, open components that are called chambers. The chambers are finite intersections of half-spaces and therefore they have faces.

To define the chambers and the faces of a hyperplane arrangement, notice that a hyperplane cuts the space into two half spaces, call one of them positive and the other one negative. A chamber can be specified by keeping track for every hyperplane of whether it is on the positive or negative half space of the hyperplane. Let m be the number of hyperplanes in \mathcal{A} . A chamber can be expressed as a vector with m coordinates, each one of them is either $+$ or $-$. A face can also be viewed as vector with m coordinates but this time the coordinates can also be zero, if the face lies on the hyperplane.

Let \mathcal{F} be the set of all faces and \mathcal{C} be the set of all chambers. The following hyperplane arrangement in \mathbb{R}^2 contains 7 chambers and 19 faces (chambers, edges and points):

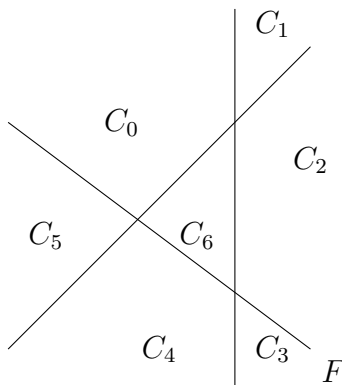


Figure 1

There is a notion of product between a face F and a chamber C . The result will be the unique chamber which is the nearest to C (in the sense of crossing the fewest number of hyperplanes) and has F as a face, in other words faces act on chambers in the above way. The product FC is called

the projection of C on F . For example, in figure 1 the product of C_0 with F is C_2 . The product of faces is defined more carefully in section 5 where it is shown to have the following associative property:

$$F(GC) = (FG)C$$

for all $F, G \in \mathcal{F}$ and $C \in \mathcal{C}$. The rigorous algebraic definition of the product is introduced in section 5.

Bidigare, Hanlon and Rockmore (BHR) [7] defined a random walk on \mathcal{C} using the above action of \mathcal{F} on \mathcal{C} and characterized its eigenvalues. Starting with a probability measure w on \mathcal{F} , a step in the walk is the following: from $C \in \mathcal{C}$, choose F according to w and move to FC . Denote by C^t the t^{th} configuration of the walk, that is the chamber the walk is on after t steps of running the process. Then,

$$C^t = F^t \dots F^2 F^1 C_0$$

where F^i denotes the face picked at time i .

Brown and Diaconis [9] proved that the transition matrix K of this Markov Chain is diagonalizable and they reproved the BHR result. They also found a necessary and sufficient condition on w so that K has a unique stationary distribution. This condition is that w separates the hyperplanes of \mathcal{A} , namely for every $H \in \mathcal{A}$ there is a face $F \not\subseteq H$ such that $w(F) > 0$. Under that assumption, they provide a stochastic description for the stationary measure π : sample without replacement from w and apply these faces in inverse order to any starting chamber (this way the first chosen face is the last to be applied). In this paper, w is assumed to be separating, so that there exists a notion of convergence to this unique distribution. Athanasiadis and Diaconis have a similar discussion in [5], but they use purely combinatorial methods as well as a coupling argument.

The approach of this paper is more probabilistic. It involves a strong stationary time argument. It thus gives stronger bounds than previous methods; bounds in separation distance, which is defined as:

$$s(t) = \max_{x_0 \in \mathcal{C}} \left(1 - \min_{x \in \mathcal{C}} \frac{K_{x_0}^{*t}(x)}{\pi(x)} \right)$$

where $K_{x_0}^{*t}(x)$ denotes the probability of starting the process at x_0 and moving to x after t steps. To state the result consider the following definition:

Definition 1. Let F, G be two faces and denote by $I_F = \{H \in \mathcal{A} : F \subset H\}$. Then F and G are called adjacent if

$$I_F = I_G$$

This way the space \mathcal{F} is partitioned in blocks B_i each one consisting only of faces adjacent to the i^{th} hyperplane. Let

$$w(B_j) = \sum_{F \in B_j} w(F)$$

then the first new result of this paper states

Theorem 2. Let \mathcal{A} be a hyperplane arrangement and w the measure on \mathcal{F} . If K is the transition matrix of the Markov Chain described above then

$$s(t) \leq \sum_j (1 - w(B_j))^t$$

where the sum is taken over all blocks of positive weight.

In particular, for the Tsetlin library described above, Theorem 2 says that

Theorem 3.

$$s(t) \leq \sum_{i=1}^n (1 - w(i))^t$$

where $w(i)$ is the weight of the i^{th} card.

Theorem 3 gives the correct answer for the mixing time if the weights are all equal to $1/n$, which is $n \log n + cn$. Yet in some cases, such as the riffle shuffles, Theorem 2 does not give such accurate answers. Section 9 examines a special case where the following symmetry condition is required: assume that a group G acts on V preserving the hyperplane arrangement \mathcal{A} so that the action restricted on the chambers is transitive. If for $F, L \in \mathcal{F}$ there is $g \in G$ such that

$$F = gL \text{ then we require that } w(F) = w(L) \tag{1}$$

In this case the result is

Theorem 4. *Under the symmetry conditions,*

$$s(t) \leq \sum_{i=1}^m \left(1 - \sum_{\substack{F \in \mathcal{F} \\ F \notin H_i}} w(F) \right)^t$$

A few important examples can be found in sections 3 and 4. Section 5 includes the setup for the strong stationary time, as well as the proof that it is a strong stationary time indeed. Finally, Section 8 gives the details of the proof of Theorem 2.

Remark 5. *This paper provides only the basic information around hyperplane arrangements that is needed for the setup of the problem and the proof of the results. The reader is encouraged to learn more about hyperplane arrangements by reading [16].*

2 Strong stationary times

The main results of this paper are proven using strong stationary times. Diaconis and Aldous [2] introduced the following definition;

Definition 6. *Fix $x_0 \in X$. A strong stationary time is a stopping time τ such that for every $A \subset X$ and $k \geq 0$ it holds that*

$$\mathbb{P}_{x_0}(X_k \in A | \tau \leq k) = \pi(A)$$

where X_k is the state that the Markov Chain is at time k .

Aldous and Diaconis [2] proved the following theorem which is the main link between strong stationary times and separation distance:

Lemma 7. *If τ is a strong stationary time then for $t > 0$,*

$$s(t) \leq \mathbb{P}(\tau > t)$$

3 Braid Arrangement and Pop shuffles.

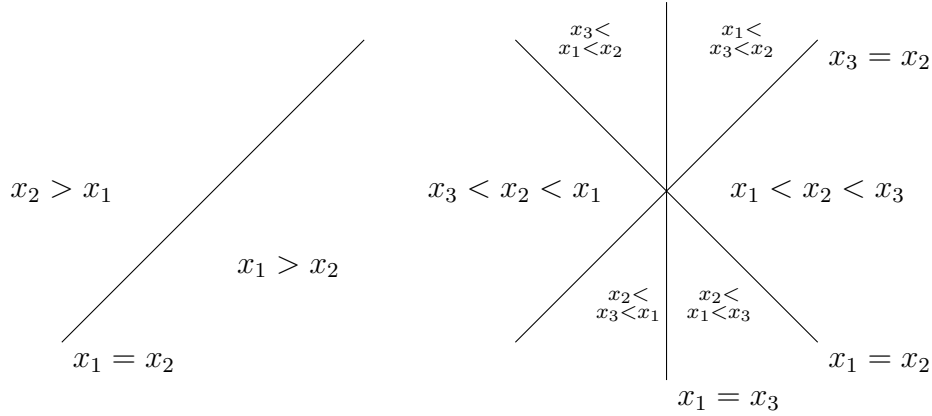
Shuffling schemes can be viewed as a Markov chain on the chambers of the braid arrangement. As presented in detail later, the chamber's of this arrangement are indexed by permutations and the faces are indexed by ordered block partitions. The following scheme is an example of such a Markov chain:

Consider all ordered block partitions of $[n] = \{1, 2, \dots, n\}$ and assign weights to them. The card shuffling suggests to pick an ordered block partition A_1, A_2, \dots, A_m according to the weights and remove from the deck the cards indicated by A_1 and put them on the top, keeping their relative order fixed. Then put the cards indicated by A_2 and put them exactly below the A_1 cards, keeping their relative order fixed and so on. This card shuffling is known as the pop shuffle.

This shuffling scheme is a Markov Chain on the chambers of a specific hyperplane arrangement. In particular consider the hyperplanes

$$x_i = x_j \quad (2)$$

The following pictures represent the braid arrangement for \mathbb{R}^2 and \mathbb{R}^3 respectively, where in \mathbb{R}^3 the lines drawn correspond to planes:



Then the chambers are in one to one correspondence with S_n . That is because in the interior of a chamber none of the coordinates are equal to each other and in fact the ordering of the coordinates is fixed. For example, the chamber that corresponds to $\sigma \in S_b$ is

$$x_{\sigma(n)} < x_{\sigma(n-1)} < \dots < x_{\sigma(1)} \quad (3)$$

The faces are exactly the ordered partitions of $[n]$, meaning that some of the coordinates are equal, forming these way blocks that are ordered. For example,

$$\{1, 2, 3\}\{4, 5\}\{6, 7, \dots, n\}$$

correspond to

$$x_1 = x_2 = x_3, x_4 = x_5, x_6 = x_7 = \dots = x_n$$

$$x_1 < x_4 < x_6$$

Therefore, for this Markov Chain Theorem 2 says that

$$s(t) \leq \sum (1 - w(B_j))^t$$

but if we add the symmetry conditions of T_3 then we might be able to get better bounds.

Weighted subsets Markov Chain. Label the cards of a deck with the numbers $1, 2 \dots n$ from top to end. Let S_i be any subset of $\{1, 2, \dots n\}$ and w_i the weight assigned to S_i for $i = 1, 2 \dots nL$, where some of the w_i are allowed to be zero. The only condition on the w_i 's is that for every $i, j \in \{1, 2, 3, \dots n\}$ there is a subset A of $1, 2 \dots n$ with positive weight such that $i \in A, j \notin A$.

The Markov chain of this section picks a subset S_j with probability w_j and then look at the deck of cards: remove the cards whose assigned number is in S_j and move them to the top of the deck keeping their previous relative order. The stationary measure is sampling without replacement according to the w_j 's and perform the sorting on the deck of cards starting from the last subset picked.

During the Markov Chain process, $i, j \in \{1, 2, 3, \dots n\}$ have been separated if at least once we have picked a subset of $\{1, 2, 3, \dots n\}$ which contains only one of i, j .

In this case, Theorem 2 says the following:

$$s(t) \leq \sum (1 - w_j)^t$$

which is proven again by the same strong stationary time argument. Let T be the first time that all subsets of $\{1, 2, \dots n\}$ with positive weight have been picked. Notice that if time T has occurred then all pairs i, j have been separated. This T is known to be a coupling time due to work of Athanasiadis and Diaconis [5].

Lemma 8. *T is a strong stationary time.*

The proof of lemma 8 is omitted since it is a special case of the proof presented in Section 6. For more details on this example, see section 4B of [5].

Inverse Riffle Shuffles. Inverse riffle shuffles, as presented by Aldous and Diaconis [2], relies on marking some of the cards with zeros and the rest with ones and then moving the former ones on top, preserving their relative order. This corresponds to sampling among the two-block ordered partitions $\{c_1, c_2, \dots, c_i\}\{[n] - \{c_1, c_2, \dots, c_i\}\}$ with weights:

$$w(B) = \begin{cases} 1/2^{n-1}, & \text{if } B = [n] \\ 1/2^n, & \text{if } B = (s, [n] \setminus s) \text{ where } s \neq \emptyset, s \neq [n] \\ 0, & \text{if otherwise} \end{cases}$$

Although Bayer and Diaconis [6] prove that the optimal upper bound for the total variation mixing time is $\frac{3}{2} \log_2 n + \theta$, yet work done by Aldous and Diaconis [2] and Assaf, Diaconis and Soundararajan [4] proves that the separation distance mixing time is $2 \log_2 n + \theta$. Several other metrics have also been studied: [4] have studied the l_∞ norm as well, while Stark, Gannesh and O'Connell [17] studied the Kullback-Leibler distance.

As discussed in Athanasiadis and Diaconis in [5] there is a generalization of this card shuffling, namely marking the cards with a number in $\{0, 1, \dots, a-1\}$ according to the multinomial distribution. Then move the ones marked with zeros on top, keeping their relative order fixed, and continue with the ones marked with 1 etc. This is a generalization of a strong stationary argument of Aldous' and Diaconis' in [2], giving an upper bound for the general inverse riffle shuffle of the form $2^{\frac{\log n}{\log a}}$.

Inverse shuffle is obviously a special case of the braid arrangement Markov chain and the weights assigned to an ordered partition are determined according to the multinomial distribution. Obviously, the only ordered partitions of positive weights are the ones that have at most a blocks. The bounds given by the strong stationary time in this paper are not great in the case of the riffle shuffles. For example in the case where $a = 2$, Theorem 5 says:

$$s(t) \leq \sum_{i=0}^n \binom{n}{i} \left(1 - \frac{1}{2^n}\right)^t = 2^n \left(1 - \frac{1}{2^n}\right)^t$$

which gives an exponential bound for the mixing time.

This is fixed by Theorem 4 since for $t = 2 \log_2 n + c$

$$s(t) \leq \sum_{i=1}^{\frac{n(n-1)}{2}} \left(1 - \frac{2^{n-1}}{2^n}\right)^t = \frac{n(n-1)}{2} \left(\frac{1}{2}\right)^t \leq \frac{1}{2^{c+1}}$$

Random to Top-Tsetlin Library. Let $w(j)$ denote the weight assigned to the j^{th} card such that $w(j) > 0$ for all $j \in \{1, 2, \dots, n\}$ and $\sum_{j=1}^n w(j) = 1$. Consider the following Markov Chain on S_n : start from a state x in S_n . With probability $w(j)$ remove card j and place it on top.

The stationary distribution is the Luce model, which has stationary distribution described as sampling from an urn with n balls without replacement, picking ball j with probability $w(j)$.

The eigenvalues of this Markov chain are known due to Phatarfod [15]. Brown and Diaconis [9], Athanasiadis and Diaconis [5] also present the eigenvalues of the Tsetlin Library as an example of a hyperplane walk. Brown [8] has analyzed the q -analogue of the Tsetlin library. In this section, a strong stationary argument is given:

Lemma 9. *For the Tsetlin Library with weights $w(i)$, let T be the first time we have touched all cards. Then T is a strong stationary time.*

Roughly, consider first of all the case where all cards have weights $1/n$. Then the first time a card is moved to the top of the deck, the top card is a random card. When a new card is moved to the top then the order between the first two cards is random. Inductively, if there are i cards on the top part of the deck with random order, given than a new, random card is moved to top will result to having $i + 1$ cards on the top part of the deck in random order. Note that if given that i cards have been touched and then one of them is chosen randomly to be moved to the top of the deck then the order of the i top cards is still random, even conditional on i and the times of moving.

Now if card c has each own weight $w(c)$ then the probability that this card is moved to the top during the first step is exactly $w(c)$. Let's see what happens when a new card is moved to the top. Then the probability that card a is on top, followed by card c which is in the second position give that exactly two cards have been touched is $\frac{w(a)w(c)}{(1-w(c))}$. Assume that the probability of having c_1 on top, c_2 on the second position, ..., c_i on the i^{th} position, given that i cards have been moved, is

$$\frac{w(c_1)w(c_2) \dots w(c_i)}{(1 - w(c_2) \dots - w(c_i)) \dots (1 - w(c_i))}. \quad (4)$$

Then given that on the next step a new card is move to the top, the probability of having c_0 on top, c_1 on the second position, ..., c_i on the $i + 1^{th}$

position is $\frac{w(c_0)w(c_1)w(c_2)\dots c_i}{(1-w(c_1))\dots(1-w(c_i))\dots(1-w(c_i))}$. While if an already touched card gets moved to the top (4) changes accordingly.

This is presented more formally in the following proof:

Proof. Let's mark the cards that we place on top. Let T_i denote the i^{th} time we mark a new card. Then the claim is that

$$\begin{aligned} & \mathbb{P}\left(X^t(1) = c_1, X^t(2) = c_2, \dots, X^t(i) = c_i \left| \begin{array}{l} T_i = t \\ c_1, \dots, c_i \\ \text{are the marked cards at time } t \end{array} \right. \right) \\ &= \frac{w(c_1)}{w(c_1) + w(c_2) + \dots w(c_i)} \frac{w(c_2)}{w(c_2) + \dots w(c_i)} \cdots \frac{w(c_{i-1})}{w(c_{i-1}) + w(c_i)} \end{aligned} \quad (5)$$

To prove this use the following inductive argument. First of all, it is clear that $T_1 = 1$ and that

$$\mathbb{P}\left(X^1(1) = c_1 \left| \begin{array}{l} T_1 = 1 \\ c_1 \text{ is the marked card at time } 1 \end{array} \right. \right) = 1$$

and

$$\begin{aligned} & \mathbb{P}\left(X^t(1) = c_1, X^t(2) = c_2 \left| \begin{array}{l} T_1 = t \\ c_1, c_2 \text{ are the marked cards at time } t \end{array} \right. \right) = \\ & \frac{w(c_1)}{w(c_1) + w(c_2)} \end{aligned}$$

Let's assume (5) and take the inductive step

$$\begin{aligned} & \mathbb{P}\left(X^t(1) = c_1, X^t(2) = c_2, \dots, X^t(i+1) = c_{i+1} \left| \begin{array}{l} T_{i+1} = t \\ c_1, \dots, c_{i+1} \\ \text{are the marked cards at time } t \end{array} \right. \right) \\ &= \mathbb{P}\left(X^t(2) = c_2, \dots, X^t(i+1) = c_{i+1} \left| \begin{array}{l} T_{i+1} = t \\ c_1, \dots, c_{i+1} \\ \text{are the marked cards at time } t \end{array} \right. \right) = \\ & \frac{w(c_1)}{w(c_1) + w(c_2) + \dots w(c_i) + w(c_{i+1})} \frac{w(c_2)}{w(c_2) + \dots w(c_i) + w(c_{i+1})} \cdots \frac{w(c_i)}{w(c_i) + w(c_{i+1})} \end{aligned}$$

The above almost finishes the proof. It remains to prove that if I move one of the marked cards then the distribution of the ordering of marked cards is

the measure described by (5). I have to check that

$$\begin{aligned} & \mathbb{P} \left(X^t(1) = c_1, X^t(2) = c_2, \dots, X^t(i) = c_i \left| \begin{array}{l} T_i = t-1 \\ c_1, \dots, c_i \\ \text{are the marked cards at time } t \end{array} \right. \right) \\ &= \frac{w(c_1)}{w(c_1) + w(c_2) + \dots + w(c_i)} \frac{w(c_2)}{w(c_2) + \dots + w(c_i)} \cdots \frac{w(c_{i-1})}{w(c_{i-1}) + w(c_i)} \end{aligned}$$

It is true because

$$\begin{aligned} & \mathbb{P} \left(X^t(1) = c_1, X^t(2) = c_2, \dots, X^t(i) = c_i \left| \begin{array}{l} T_i = t-1, T_{i+1} > t \\ c_1, \dots, c_i \\ \text{are the marked cards at time } t \end{array} \right. \right) = \\ & \mathbb{P} \left(X^t(1) = c_1, X^t(2) = c_2, \dots, X^t(i) = c_i \left| \begin{array}{l} T_i = t-1, T_{i+1} > t \\ c_1, \dots, c_i \\ \text{are the marked cards at time } t \\ X^{t-1}(1) = c_1, X^{t-1}(2) = c_2, \\ \dots, X^{t-1}(i) = c_i \end{array} \right. \right) \\ & \mathbb{P} \left(X^{t-1}(1) = c_1, X^{t-1}(2) = c_2, \dots, X^{t-1}(i) = c_i \left| \begin{array}{l} T_i = t-1, T_{i+1} > t \\ c_1, \dots, c_i \\ \text{are the marked cards at time } t \end{array} \right. \right) \\ & + \sum_{j=2}^i \mathbb{P} \left(X^t(1) = c_1, X^t(2) = c_2, \dots, X^t(i) = c_i \left| \begin{array}{l} T_i = t-1, T_{i+1} > t \\ c_1, \dots, c_i \\ \text{are the marked cards at time } t \\ X^{t-1}(j) = c_1, X^{t-1}(1) = c_2, \\ \dots, X^{t-1}(i-1) = c_i \end{array} \right. \right) \\ & \mathbb{P} \left(X^{t-1}(j) = c_1, X^{t-1}(1) = c_2, \dots, X^{t-1}(i-1) = c_i \left| \begin{array}{l} T_i = t-1, T_{i+1} > t \\ c_1, \dots, c_i \\ \text{are the marked} \\ \text{cards at time } t \end{array} \right. \right) = \\ & \frac{w(c_1)}{w(c_1) + w(c_2) + \dots + w(c_i)} \\ & \left(\mathbb{P} \left(X^{t-1}(1) = c_1, X^{t-1}(2) = c_2, \dots, X^{t-1}(i) = c_i \left| \begin{array}{l} T_i = t-1, T_{i+1} > t \\ c_1, \dots, c_i \\ \text{are the marked cards at time } t \end{array} \right. \right) + \right. \end{aligned}$$

$$\sum_{j=2}^i \mathbb{P} \left(X^{t-1}(j) = c_1, X^{t-1}(1) = c_2, \dots, X^{t-1}(i-1) = c_i \left| \begin{array}{l} T_i = t-1, T_{i+1} > t \\ c_1, \dots, c_i \\ \text{are the marked} \\ \text{cards at time } t \end{array} \right. \right) =$$

$$\frac{w(c_1)}{w(c_1) + w(c_2) + \dots + w(c_i)} \frac{w(c_2)}{w(c_2) + \dots + w(c_i)} \frac{w(c_{j-1})}{w(c_{j-1}) + w(c_j)}$$

□

To prove Theorem 3 will simply use a union bound:

Proof. Let A_i^t be the event that at t steps I haven't touched card c_i . Then we have that

$$P(T > t) \leq P(\cup_{i=1}^n A_i^t) \leq \sum_{i=1}^n (1 - w_i)^t$$

□

Random to top or bottom. Consider the card shuffling where a card is chosen at random and is moved to the top or the bottom with probability $1/2$. This is again a random walk on the chambers of the braid arrangement. The faces used are of the form $\{\{c\}, \{[n] \setminus \{c\}\}\}$ and $\{\{[n] \setminus \{c\}\}, \{c\}\}$ each one having weight $1/2n$. Theorem 5 says that if $t = n \log n + cn$ then

$$s(t) \leq n \left(1 - \frac{1}{n}\right)^t \leq e^{-c}$$

For the weighted version of the card shuffling let w_c^+ denote the weight of $\{\{c\}, \{[n] \setminus \{c\}\}\}$ and w_c^- denote of $\{\{[n] \setminus \{c\}\}, \{c\}\}$. Then theorem 5 gives that

$$s(t) \leq \sum_{c=1}^n (1 - w_c^- - w_c^+)^t$$

To finish off and get explicit bounds in the weighted cases depends a lot on the weights $w(i)$. For some sample calculations see the work of Diaconis [10].

4 The Boolean Arrangement.

The Boolean arrangement consists simply of the hypeplanes $x_i = 0$, $1 \leq i \leq n$ in \mathbb{R}^n . Each chamber is specified by the sign of its coordinates, in other words they are the 2^n orthants in \mathbb{R}^n . The faces are in bijection with $\{-, 0, +\}^n$. The projection FC of a chamber C on a face F is a chamber who adopts all the signs non-zero coordinates of F an the rest of the coordinates have the signs of C .

Neighborhood walk on the hypercube. The chambers of the Boolean arrangement are as explained above in a bijection with $\{-, +\}^n$, in other words each chamber corresponds to a vertex of the n -dimensional hypercube. If the only positive weighted faces are the E_i^\pm , whose i^{th} coordinate is \pm and the rest are zero, then the Markov Chain corresponds to the weighted nearest neighbor random walk on the hypercube, which corresponds to choosing a coordinate and switching it to \pm . Denote the weight of E_i^\pm by w_i^\pm . The transition matrix in this case is

$$K(x, x') = \begin{cases} \sum_{i=1}^n w_i^{x_i}, & \text{if } x = x' \\ w_i^{-x_i}, & \text{if } x \text{ is obtained from } x' \\ & \text{by switching the } i^{th} \text{ coordinate of } x \\ 0, & \text{otherwise.} \end{cases}$$

A strong stationary time in this case is the first time that all coordinates have been picked. Then

$$s(t) \leq \sum_{i=1}^n (i - w_i^+)t + \sum_{i=1}^n (i - w_i^-)t$$

In particular, for the case where $w_i^\pm = \frac{1}{2n}$ the upper bound for the separation time mixing time will be bounded by $2n \log 2n + cn$, but theorem 4 improves the bound to $n \log n + cn$. It is straightforward to show a lower bound of the form $n \log n - cn$ for separation distance. This random walk is very well studied: Aldous [1] and Diaconis and Shashahani [11] have proved the cut-off for the total variation distance mixing time at $\frac{n}{2} \log n + cn$ using Fourier analysis.

A non-local walk on the hypercube. Consider the following walk on the hypercube: fix $k \geq 1$ and pick k coordinates at random and flip a fair

coin for each one of them to determine whether to turn them into ones or zeros. In this case, Theorem 4 gives an upper bound of the form $\frac{n}{k} \log n + c \frac{n}{k}$, since:

$$s(t) \leq \sum_{i=1}^n \left(1 - \frac{\binom{n-1}{k-1}}{\binom{n}{k}}\right)^t = \sum_{i=1}^n \left(1 - \frac{k}{n}\right)^t$$

A walk on a finite, transitive graph. As described at the beginning of this paper, a special case of a hyperplane arrangement walk could be the following process on a finite, transitive graph: pick a vertex at random and color it and its neighbors all red or blue with probability 1/2. Then the strong stationary time suggests to stop once at least one representative of each neighborhood has been picked. If S is a minimum vertex cover then theorem 4 says that

$$s(t) \leq |S| \left(1 - \frac{1}{n}\right)^t$$

Remark 10. *The described process can be considered for any type of graph and the coupling bound of Athanasiadis and Diaconis works. Yet to pass to separation distance and use Theorem 4 transitivity is needed because of the symmetry conditions.*

5 Preliminaries

In the following sections, three strong stationary time arguments will be analyzed for the general hyperplane arrangement Markov chain. The easiest one to state is the first time all positively weighted faces have been picked. Call this time T_1 . The second strong stationary time T_2 is very similar to T_1 , so more details can be found in section 7. The third strong stationary time, which will be called T_3 is the first time that the $F_{i_1} F_{i_2} \dots F_{i_l}$ is a chamber, given that F_{i_j} is the face picked at time j . T_3 gives much better bounds than T_1 and T_2 .

This primary section proves a useful geometric lemma that will simplify the explanation that the above stopping times are indeed strong stationary times. Moreover, the following definition is the key fact behind that lemma. By definition, any face F can be written in the following form:

$$F = \cap_{i \in I} H_i^{\sigma_i(F)}$$

where $\sigma_i(F) \in \{+, -, 0\}$, H_i^+ corresponds to the right open half-space determined by H_i (and respectively H_i^- for the left one) and $H_i^0 = H_i$. Notice that if $\sigma_i(F) \neq 0$ for all i if and only if F is a chamber. The faces form a semigroup under the following product:

Definition 11. *If F, G are two faces then*

$$FG = \cap_{i \in I} H_i^{\sigma_i(FG)}$$

where

$$\sigma_i(FG) = \begin{cases} \sigma_i(F), & \text{if } \sigma_i(F) \neq 0 \\ \sigma_i(G), & \text{otherwise} \end{cases}$$

It turns out that multiplication of faces satisfies both the “idempotence” the “deletion property”, that is if F and G are two faces then

$$F \cdot F = F \text{ and } FGF = FG \quad (6)$$

A property of this type has already appeared in special types of semigroups called left-regular bands. Brown [8] has also used this property to find the eigenvalues of a similar Markov chain on semigroups. The “deletion property” leads to the following lemma:

Lemma 12. *Let $F_{i_j} \in \{F \in \mathcal{F} : w(F) > 0\}$, then*

$$F_{i_a} F_{i_k} F_{i_{k-1}} \dots F_{i_{a+1}} F_{i_a} F_{i_{a-1}} \dots F_{i_1} = F_{i_a} F_{i_k} F_{i_{k-1}} \dots F_{i_{a+1}} F_{i_{a-1}} \dots F_{i_1} \quad (7)$$

In other words, if the left term of a product of faces has appeared in the product earlier, then it can be omitted from all the positions but the most left one and the product will remain the same. Also, let T_i denote the i^{th} time a new face is picked then

$$\mathbb{P} \left(C^t = C \mid \begin{array}{l} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \\ \text{and it is the } l+1 \text{ time it has been picked} \end{array} \right) =$$

$$\mathbb{P} \left(C^{t-l} = C \left| \begin{array}{l} T_i = t-l \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right) \quad (8)$$

for all t that the condition could be applied to.

Proof. To prove that equation (7) holds it suffices to check the deletion property described by equation (6). This is easy because

$$\sigma_i(F \cdot F) = \sigma_i(F) \text{ and } \sigma(FGF) = \sigma(FG)$$

Equation 8 holds because if

$$C^t = F_j p^{t-1}(F_1, F_2, \dots F_i) C_0$$

where $p^{t-1}(F_1, F_2, \dots F_i)$ is a product of $F_1, F_2, \dots F_i$ of length $t-1$ where F_j appears exactly l times and non of these terms is omitted and if

$$C^{t-l} = F_j p^{t-1}(F_1, F_2, \dots F_{j-1}, \emptyset, F_{j+1} \dots F_i) C_0$$

then

$$C^t = C^{t-l}$$

and thus

$$\begin{aligned} & \mathbb{P} \left(C^t = C \left| \begin{array}{l} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \\ \text{and it is the } l+1 \text{ time it has been picked} \end{array} \right. \right) = \\ & \frac{1}{\binom{t-1}{l}} \sum_{1 \leq m_1 < m_2 < \dots < m_l < t} \mathbb{P} \left(C^t = C \left| \begin{array}{l} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \\ \text{and it is the } l+1 \\ \text{time it has been picked,} \\ m_1, m_2, \dots m_l \\ \text{are the moments that we picked } F_j \end{array} \right. \right) = \\ & \frac{1}{\binom{t-1}{l}} \sum_{1 \leq m_1 < m_2 < \dots < m_l < t} \mathbb{P} \left(C^{t-l} = C \left| \begin{array}{l} T_i = t-l \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right) = \end{aligned}$$

$$\mathbb{P} \left(C^{t-l} = C \mid \begin{array}{l} T_i = t - l \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right)$$

□

6 A first strong stationary argument

Let \mathcal{A} be a hyperplane arrangement with face weights w_F , $F \in \mathcal{F}$. Assume that w_F are separating. Let T_1 be the first time all positively weighted faces have been picked.

Lemma 13. *T_1 is a strong stationary time for the Markov chain on \mathcal{C} .*

Roughly, if face F_1 has each own weight $w(F_1)$ then the probability that F_1 is picked during the first step is exactly $w(F_1)$. Let's see what happens when a new face is picked. Then the probability that face F_2 is picked, after the last time F_1 was picked, given that exactly two faces have been touched is $\frac{w(F_2)w(F_1)}{(1-w(F_1))}$. Assume that the probability of having picked F_1 last, F_2 before that, \dots , F_i was the first face to be picked, given that i faces have been picked, is

$$\frac{w(F_1)w(F_2) \dots w(F_i)}{(1 - w(F_2) \dots - w(F_i)) \dots (1 - w(F_i))}. \quad (9)$$

Then given that on the next step a new face is picked, the probability of having picked F_0 last, F_1 before that, \dots , F_i firstly is $\frac{w(F_0)w(F_1)w(F_2) \dots w(F_i)}{(1-w(F_1)) \dots - w(F_i)) \dots (1-w(F_i))}$. While if an already touched face gets picked (9) changes accordingly.

This is presented more formally in the following proof:

Proof. The proof of the lemma is based on induction. Every time a new face is picked it will be marked.

Claim: Let T_i denote the i^{th} time a new face is marked. Then for any $F_1, F_2, \dots, F_i \in \mathbb{F}$ the following is true for all t that make the condition pos-

sible:

$$\begin{aligned}
& \mathbb{P} \left(C^t = C \mid \begin{array}{l} T_i = t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right) = \\
& \mathbb{P} \left(C^t = C \mid \begin{array}{l} T_{i-1} < t < T_i \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right) \\
& = w_{F_1, \dots, F_i}(C)
\end{aligned}$$

where $w_{F_1, \dots, F_i}(C)$ is sampling without replacement from $\{F_1, F_2 \dots F_i\}$ and applying the faces picked to C_0 in the reserve order, keeping track only of the products that will end up in C .

It is clear that if the claim is true then T_1 is a strong stationary time. First of all,

$$\mathbb{P}(C^t = C \mid \text{only set } F \text{ is marked at time } t) = \begin{cases} 1, & \text{if } FC_0 = C \\ 0, & \text{otherwise} \end{cases}$$

In order to do induction on i , assume that for any $F_1, F_2, \dots, F_{i-1} \in \mathbb{F}$ the following is true for all t that make the condition possible:

$$\begin{aligned}
& \mathbb{P} \left(C^t = C \mid \begin{array}{l} T_{i-1} = t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_{i-1} \end{array} \right) = \\
& \mathbb{P} \left(C^t = C \mid \begin{array}{l} T_{i-1} < t < T_i \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_{i-1} \end{array} \right) \\
& = w_{F_1, \dots, F_{i-1}}(C)
\end{aligned}$$

where $w_{F_1, \dots, F_{i-1}}(C)$ is sampling without replacement from $\{F_1, F_2 \dots F_{i-1}\}$ and applying the faces picked to C_0 in the reserve order, keeping track only of the products that will end up in C .

Then

$$\begin{aligned}
& \mathbb{P} \left(C^t = C \mid \begin{array}{c} T_i = t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right) = \\
& \sum_{j=1}^i \mathbb{P} \left(C^t = C \mid \begin{array}{c} T_i = t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ \text{face } F_j \text{ was chosen at time } t \end{array} \right) \\
& \mathbb{P} \left(\text{face } F_j \text{ was chosen at time } t \mid \begin{array}{c} T_i = t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right) = \\
& \sum_{j=1}^i \frac{w(F_j)}{w(F_1) + w(F_2) + \dots + w(F_i)} \mathbb{P} \left(C^t = C \mid \begin{array}{c} T_{i-1} < t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ \text{face } F_j \text{ was chosen at time } t \end{array} \right) \\
& \hspace{15em} (10)
\end{aligned}$$

The induction step will help to prove that (10) is equal to $w_{F_1, \dots, F_i}(C)$. The previous step configuration C^{t-1} will run through all possible chambers \overline{C} and therefore

$$\begin{aligned}
& \mathbb{P} \left(C^t = C \mid \begin{array}{c} T_{i-1} < t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ \text{face } F_j \text{ was chosen at time } t \end{array} \right) = \\
& \sum_{\overline{C}: F_j \overline{C} = C} \mathbb{P} \left(C^{t-1} = \overline{C} \mid \begin{array}{c} T_{i-1} < t \\ \text{the marked faces at time } t-1 \text{ are} \\ F_1, F_2, \dots, F_{i-1} \end{array} \right) = \\
& \sum_{\overline{C}: F_j \overline{C} = C} w_{F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_i}(\overline{C})
\end{aligned}$$

Thus

$$\begin{aligned}
& \mathbb{P} \left(C^t = C \left| \begin{array}{c} T_i = t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right. \right) = \\
& \sum_{j=1}^i \sum_{\bar{C}: F_j \bar{C} = C} \frac{w(F_j)}{w(F_1) + w(F_2) + \dots + w(F_i)} w_{F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_i}(\bar{C}) = \\
& w_{F_1, \dots, F_i}(C)
\end{aligned}$$

To complete the proof we need to also prove that the following measure also coincides with $w_{F_1, \dots, F_i}(C)$.

$$\begin{aligned}
& \mathbb{P} \left(C^t = C \left| \begin{array}{c} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right. \right) = \\
& \sum_{j=1}^i \frac{w(F_j)}{w(F_1) + w(F_2) + \dots + w(F_i)} \mathbb{P} \left(C^t = C \left| \begin{array}{c} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right)
\end{aligned}$$

But equation (8) says that

$$\begin{aligned}
& \mathbb{P} \left(C^t = C \left| \begin{array}{c} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right) = \\
& \sum_{l=0}^{t-i} \mathbb{P} \left(C^t = C \left| \begin{array}{c} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \\ F_j \text{ has been picked } l+1 \text{ times} \end{array} \right. \right)
\end{aligned}$$

$$\mathbb{P} \left(F_j \text{ has been picked } l+1 \text{ times} \left| \begin{array}{c} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right) =$$

$$\sum_{l=0}^{t-i} \mathbb{P} \left(C^{t-l} = C \left| \begin{array}{l} T_i = t-l \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t-l \end{array} \right. \right)$$

$$\mathbb{P} \left(F_j \text{ has been picked } l+1 \text{ times} \left| \begin{array}{l} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right) =$$

$$w_{F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_i}(C) \sum_{l=0}^{t-i} \mathbb{P} \left(F_j \text{ has been} \left| \begin{array}{l} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right) =$$

$$w_{F_1, \dots, F_{j-1}, F_{j+1}, \dots, F_i}(C)$$

The last equality is because of induction. Therefore, indeed

$$\mathbb{P} \left(C^t = C \left| \begin{array}{l} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right. \right) =$$

$$\sum_{j=1}^i \frac{w(F_j)}{w(F_1) + w(F_2) + \dots + w(F_i)} \mathbb{P} \left(C^t = C \left| \begin{array}{l} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \\ F_j \text{ was picked at time } t \end{array} \right. \right)$$

$$= w_{F_1, \dots, F_i}(C)$$

and this leads to the proof of the claim. \square

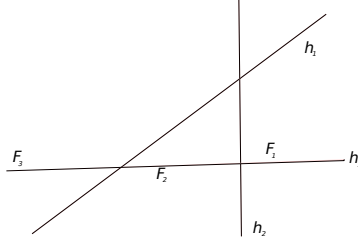
7 An improved strong stationary time

This section introduces a strong stationary time for a special case of the Markov chain on hyperplane arrangements. Consider the following definition:

Definition 14. Let F, G be two faces and denote by $I_F = \{H_i \in \mathcal{A} : \sigma_i(F) \neq 0\}$. Then F and G are called adjacent if

$$I_F = I_G$$

For example, in the following picture F_1, F_2 and F_3 are adjacent since $F_1 = (+, +, 0)$, $F_2 = (+, -, 0)$ and $F_3 = (-, -, 0)$ if the coordinates are taken according to h_1, h_2, h_3 in order. More specifically $I_{F_1} = I_{F_2} = I_{F_3} = \{h_1, h_2\}$.



Adjacency is an equivalence relationship and therefore there is a partition of the faces on blocks where each block contains adjacent faces. Each block is determined by the positions of the non-zero coordinates. The weight of the block B_j is

$$w(B_j) = \sum_{F \in B_j} w(F)$$

Let T_2 be the first time that at least one representative of each positive weighted block been picked. The proof of the fact that T_2 is a strong stationary time depends on a second version of the stationary measure, that is introduced in Brown's and Diaconis' section 3 [9]:

Remark 15. The stationary measure π is the same as sampling faces F_1, \dots, F_l with replacement until the outcome of $F_1 F_2 \dots F_l$ is a chamber.

A similar description for the stationary measure is the following:

Lemma 16. The stationary measure is the same as sampling with replacement from the faces and stop sampling the first time that at least one representative from each block has been picked.

Proof. Let q be the occurring measure when sampling without replacement from the faces, multiplying in the reverse order and stopping when all blocks have been represented. The goal is to prove that $q = \pi$. The idea is that unnecessary terms can be omitted.

Let $C \in \mathcal{C}$ and $F_{i_1}, F_{i_2}, \dots, F_{i_j}, \dots, F_{i_l}$ be an ordering of the faces so that $C = F_{i_1} F_{i_2} \dots F_{i_j} \dots F_{i_l}$ and $F_{i_1}, F_{i_2}, \dots, F_{i_j}$ is such so that every block is represented but not every block is represented in $F_{i_1}, F_{i_2}, \dots, F_{i_{j-1}}$. Then the following term appears $\pi(C)$ as a summand:

$$\sum_{\sigma \in S_{l-j}} \mathbb{P} \left(\begin{array}{c} F_{i_1} \text{ was the first face picked} \\ \vdots \\ F_{i_j} \text{ was the } j^{th} \text{ face picked} \\ \vdots \\ F_{\sigma(i_l)} \text{ was the last face picked} \end{array} \right) =$$

$$\sum_{\sigma \in S_{l-j}} \mathbb{P} \left(\begin{array}{c|c} F_{\sigma(i_{j+1})} \text{ was the } j+1 \text{ face picked} & F_{i_1} \text{ was the first face picked} \\ \vdots & \vdots \\ F_{\sigma(i_l)} \text{ was the last face picked} & F_{i_j} \text{ was the } j^{th} \text{ face picked} \end{array} \right)$$

$$\mathbb{P} \left(\begin{array}{c} F_{i_1} \text{ was the first face picked} \\ \vdots \\ F_{i_j} \text{ was the } j^{th} \text{ face picked} \end{array} \right) = \mathbb{P} \left(\begin{array}{c} F_{i_1} \text{ was the first face picked} \\ \vdots \\ F_{i_j} \text{ was the } j^{th} \text{ face picked} \end{array} \right)$$

where $\mathbb{P} \left(\begin{array}{c} F_{i_1} \text{ was the first face picked} \\ \vdots \\ F_{i_j} \text{ was the } j^{th} \text{ face picked} \end{array} \right)$ appears as a summand in $q(C)$.

This justifies why $\pi = q$. \square

Lemma 17. T_2 is a strong stationary time.

Proof. The goal is to prove that

$$\mathbb{P}(C^t = C \mid T_2 = t) = \pi(C)$$

Let $w_{F_1, \dots, F_i}(C)$ be sampling without replacement from $\{F_1, F_2, \dots, F_i\}$ and applying the faces picked to C_0 in the reserve order, keeping track only of the products that will end up in C , just like in the proof of Lemma 13. Once more let T_i be the first time a new face is picked. The following relation is proved in the same way as Lemma 13:

$$\mathbb{P} \left(C^t = C \mid \begin{array}{c} T_i < t < T_{i+1} \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right) = \mathbb{P} \left(C^t = C \mid \begin{array}{c} T_i = t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right)$$

$$= w_{F_1, \dots, F_i}(C)$$

Remark 16 makes it clear that summing over all F_1, F_2, \dots, F_i are such so that for every block has a representative among the F_j 's then the following holds

$$\pi(C) = \sum_{F_1, F_2, \dots, F_i} w_{F_1, \dots, F_i}(C)$$

Therefore, if F_1, F_2, \dots, F_i are such so that every block has a representative among them and if at every step the face picked is marked then

$$\begin{aligned} & \mathbb{P} \left(C^t = C \mid \begin{array}{l} T_2 \leq t \\ \text{the marked faces are} \\ F_1, F_2, \dots, F_i \end{array} \right) \\ &= w_{F_1, \dots, F_i}(C) \end{aligned}$$

which completes the proof. □

8 Proof of Theorem 2.

Proof. The goal is to bound the right hand side of Lemma 7, which stated that

$$s(t) \leq P(T_2 > t)$$

T_2 gives better bounds (or constants) than T_1 therefore it is preferable to use it over T_2 . The following union bound argument will help with bounding $P(T_2 > t)$. More precisely, let A_i^t be the event that at t steps block B_i hasn't been picked. Also remember the notation $w(B_j) = \sum_{F \in B_j} w(F)$. Then we have that

$$P(T_2 > t) \leq P(\cup_{i=1}^m A_i^t) \leq \sum_{i=1}^m (1 - w(B_i))^t$$

which finishes the proof of Theorem 2. □

9 A more specialized, faster strong stationary time

Let T_3 is the first time that the product of faces picked is a chamber. According to Athanasiadis and Diaconis this is a coupling time [5]. Assuming some symmetry conditions, this is also a strong stationary time.

In this section, assume that a group G acts on V preserving the hyperplane arrangement \mathcal{A} so that the action restricted on the chambers is transitive. Assume the symmetry conditions described by equation 1. The first lemma concerns the stationary distribution:

Lemma 18. *Under the symmetry conditions the stationary measure is the uniform measure on the chambers.*

Proof. The fact that the stationary distribution π is sampling without replacement until the product of the faces picked is a chamber will be the main key to the proof of lemma 19. Sample without replacement from the faces, apply the faces to C_0 in the reverse order and let T_3 denote once more the first time that the product of the faces picked is a chamber. Let a be the number of chambers, then,

$$\pi(C) = \sum_l \sum_{F_i \neq F_j} \mathbb{P}(F_l F_{l-1} \dots F_1 C_0 = C \mid T_3 = l) \mathbb{P}(T_3 = l) = \frac{1}{a}$$

which is true because

$$\frac{\mathbb{P}(F_l F_{l-1} \dots F_1 = C \mid T = l) = \sum_{\substack{F_{i_1} F_{i_2} \dots F_{i_l} = C \\ F_{i_1} F_{i_2} \dots F_{i_{l-1}} \notin \mathcal{C} \\ F_i \neq F_j}} \mathbb{P} \left(\begin{array}{c} F_{i_1}, F_{i_2}, \dots, F_{i_l} \\ \text{are picked when sampling} \\ \text{without replacement} \\ l \text{ times} \end{array} \right)}{\sum_{D \in \mathcal{C}} \sum_{\substack{F_{i_1} F_{i_2} \dots F_{i_l} = D \\ F_{i_1} F_{i_2} \dots F_{i_{l-1}} \notin \mathcal{C} \\ F_i \neq F_j}} \mathbb{P} \left(\begin{array}{c} F_{i_1}, F_{i_2}, \dots, F_{i_l} \\ \text{are picked when sampling} \\ \text{without replacement} \\ l \text{ times} \end{array} \right)} = \frac{1}{a}$$

because of the symmetry conditions. \square

Lemma 19. *If the symmetry conditions hold then T_3 is a strong stationary time.*

Proof. Let a be the number of chambers of \mathcal{A} . To prove that

$$\mathbb{P}(C^t = C \mid T_3 = t) = \pi(C)$$

consider at first the case $t = 1$ and remember that $w(c)$ denotes the weight of a chamber C when viewed as a face.

$$\mathbb{P}(C^1 = C \mid T_3 = 1) = \frac{w(C)}{\sum_{D \in \mathcal{C}} w(D)} = \frac{1}{a}$$

where a is the number of chambers. But then because of the symmetry condition:

$$\mathbb{P}(C^2 = C \mid T_3 \leq 2) = \frac{\sum_{F_{i_1} F_{i_2} = C} w(F_{i_1}) w(F_{i_2})}{\sum_{D \in \mathcal{C}} \sum_{F_{i_1} F_{i_2} = D} w(F_{i_1}) w(F_{i_2})} = \frac{1}{a}$$

and inductively, because of the weight invariant action of G .

$$\mathbb{P}(C^t = C \mid T_3 = t) = \frac{1}{a}$$

□

Remark 20. *The strong stationary time of this section is not a strong stationary time if the symmetry conditions do not hold. To see this,*

$$\mathbb{P}(C^1 = C \mid T_3 = 1) = \frac{w(C)}{\sum_{D \in \mathcal{C}} w(D)}$$

which is not necessarily equal to $\pi(C)$.

10 Proof of Theorem 4

Proof. In this section, we assume that a group G acts on V preserving the hyperplane arrangement \mathcal{A} so that the action restricted on the chambers is transitive. Assume the symmetry conditions described by equation 1.

In this case we saw that the first time that the product of the faces picked is a strong stationary time. To bound the separation distance consider another union bound.

$$P(T_3 > t) \leq \sum_{i=1}^m \left(1 - \sum_{\substack{F \in \mathcal{F} \\ \sigma(F) \neq 0}} w(F) \right)^t$$

□

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